

# A SPLITTING THEOREM ON TORIC VARIETIES

HONGNIAN HUANG

**ABSTRACT.** Using the short time existence of the Calabi flow, we prove that any extremal Kähler metric  $\omega_E$  on a product toric variety  $X_1 \times X_2$  is a product extremal Kähler metric.

In [1], the authors proposed the following problem:

**Problem 0.1.** *Let  $X_i, i = 1, 2$  be two Kähler manifolds with Kähler classes  $[\omega_i]$ . Suppose  $\omega_E$  is an extremal Kähler metric in the Kähler class  $[\omega_1 + \omega_2]$ . Can we conclude that  $\omega_E$  is a product metric, i.e.,  $\omega_E = \omega_{E,1} + \omega_{E,2}$  where  $\omega_{E,i}$  is an extremal Kähler metric in  $[\omega_i]$ .*

In this short note, we solve the problem in the case of toric varieties.

**Theorem 0.2.** *If in addition  $X_i$  are toric varieties, then  $\omega_E$  is a product metric.*

Let  $X_i, i = 1, 2$  be toric varieties with Kähler classes  $[\omega_i]$ . Let  $P_i$  be the corresponding Delzant polytopes. Then the corresponding polytope  $P = P_1 \times P_2$ . In the symplectic side, we have symplectic potentials  $u_i$  satisfying Guillemin boundary conditions of  $P_i$ . We let  $x$  be the variable of  $P_1$  and  $y$  be variable of  $P_2$ . Our assumption shows that there exists a symplectic potential  $u$  on  $P$  and

$$u(x, y) = u_1(x) + u_2(y) + f(x, y), \quad f(x, y) \in C^\infty(\bar{P})$$

such that the scalar curvature of  $u(x, y)$  is an affine function. Our goal is to show that  $f(x, y)$  is separable. Let

$$f_1(x) = \frac{1}{\text{vol}(P_2)} \int_{P_2} f(x, y) \, dy, \quad f_2(y) = \frac{1}{\text{vol}(P_1)} \int_{P_1} f(x, y) \, dx.$$

Then we have

**Proposition 0.3.**  *$v(x, y) = u_1(x) + u_2(y) + f_1(x) + f_2(y)$  is a symplectic potential of  $P$  satisfying the Guillemin boundary conditions.*

*Proof.* It is easy to see that  $f_1(x) + f_2(y)$  is a smooth function on  $\bar{P}$ . Thus we only need to show that  $(D^2v)$  is a positive matrix in order to prove that  $v$  is a symplectic potential. To show  $(D^2v) > 0$  is equivalent to show that  $(D^2(u_1(x) + f_1(x))) > 0$  and  $(D^2(u_2(y) + f_2(y))) > 0$ . However,  $(D^2(u_1(x) + f_1(x))) > 0$  and  $(D^2(u_2(y) + f_2(y))) > 0$  just follow from the fact that  $(D^2u) > 0$ .  $\square$

Let  $\mathcal{H}$  be the set of all symplectic potentials. We define a subset of  $\mathcal{H}$ .

---

The author would like to thank Vestislav Apostolov, Paul Gauduchon and Gábor Székelyhidi for stimulating discussions. His research is financially supported by the Fondation mathématique Jacques Hadamard, Paris, France.

**Definition 0.4.**

$$\begin{aligned} \mathcal{M} = & \{ \underline{u}(x, y) \in \mathcal{H} \mid \underline{u}(x, y) = u_1(x) + u_2(y) + g_1(x) + g_2(y) \text{ s.t.} \\ & g_1(x) \in C^\infty(\bar{P}_1), \int_{P_1} f_1(x) dx = \int_{P_1} g_1(x) dx, \\ & g_2(y) \in C^\infty(\bar{P}_2), \int_{P_2} f_2(y) dy = \int_{P_2} g_2(y) dy \}. \end{aligned}$$

Then we have

**Proposition 0.5.**

$$(1) \int_P (u(x, y) - v(x, y))^2 dx dy = \min_{\underline{u} \in \mathcal{M}} \int_P (u(x, y) - \underline{u}(x, y))^2 dx dy.$$

*Proof.* (1) is equivalent to show that

$$\int_P (f(x, y) - f_1(x) - f_2(y))^2 dx dy \leq \int_P (f(x, y) - g_1(x) - g_2(y))^2 dx dy.$$

Expressing it out, we have

$$\begin{aligned} & \int_P -2f(x, y)(f_1(x) + f_2(y)) + f_1^2(x) + f_2^2(y) dx dy \\ & \leq \int_P -2f(x, y)(g_1(x) + g_2(y)) + g_1^2(x) + g_2^2(y) dx dy \end{aligned}$$

which is equivalent to

$$0 \leq \int_P (f_1(x) - g_1(x))^2 + (f_2(y) - g_2(y))^2 dx dy.$$

The equality holds iff  $f_1(x) = g_1(x)$  and  $f_2(y) = g_2(y)$ .  $\square$

*Proof of Theorem (0.2).* We use the Calabi flow to show that  $v$  is an extremal symplectic potential. We need the short time existence of the Calabi flow due to Chen and He [3] and the modified version of the Calabi flow in [4]. Let  $u(t)$  be a sequence of symplectic potentials satisfying the modified Calabi flow equation on  $P$  and  $u(0) = v$ . By Calabi and Chen's result [2], we have

$$\frac{d}{dt} \int_P (u(t) - u)^2 dx dy \leq 0.$$

Since  $u(t) \in \mathcal{M}$ , we obtain  $u(t) = v$ . It shows that  $v$  is a separable extremal symplectic potential on  $P$ . It is easy to conclude that  $f(x, y)$  is separable.  $\square$

## REFERENCES

- [1] V. Apostolov, H.N. Huang, *A splitting theorem for extremal Kähler metrics*, preprint.
- [2] E. Calabi, X.X. Chen, *Space of Kähler metrics and Calabi flow*, J. Differential Geom. 61 (2002), no. 2, 173-193.
- [3] X.X. Chen, W.Y. He, *On the Calabi flow*, Amer. J. Math. 130 (2008), no. 2, 539-570.
- [4] H.N. Huang, K. Zheng, *Stability of Calabi flow near an extremal metric*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 11 (2012), no. 1, 167-175.

HONGNIAN HUANG, CMLS, ÉCOLE POLYTECHNIQUE, 91128, PALAISEAU, FRANCE  
*E-mail address:* hnhuang@gmail.com